Lesson 3: Solving Linear Equations.

We learned about **the principle of balance** in the last lesson. In this lesson we will be using it to solve linear equations, that is, we will be finding the numbers that when put into the equation for the variable, makes the equation true. A **linear equation** is an equation of **degree one**. This means, that the highest power(exponent) of the variable in the equation is one.

Example 1: The following equations are linear equations;

$$2x = 1; \qquad 3y - 6 = 2y + 5; \qquad \frac{1}{x+1} = 2;\ x \neq -1$$

Notice that the highest exponent on the variable of each equation is 1. The equation $x^2 + 4x - 5 = 0$ **is not a linear** equation because, the highest power(exponent) of the variable x is 2.

A word about notation: When we have $2x$ this means 2 times x, or $2 \bullet x$. This is also symbolized by parentheses, $2(x)$ also means you multiply x by 2. Also, $2(x + 1)$ means you multiply each item in the parentheses by 2, that is $2(x + 1) = 2x + 2$.

Looking Ahead: The equation $x^2 + 4x - 5 = 0$ is called a Quadratic Equation because the highest power of the variable x is 2. This is also called a Polynomial of degree two. We will be seeing these in Lesson 6, and their graphs in Lesson 17.

The **general form** for a linear equation is $\mathbf{a}x + \mathbf{b} = \mathbf{0}$, where **a** and **b** are any numbers **and** $\mathbf{a} \neq \mathbf{0}$. We use the word **term** to describe the numbers and variables in an equation. The $\mathbf{a}x$, **b** and **0** are the terms of the equation $\mathbf{a}x + \mathbf{b} = \mathbf{0}$.

Now, let us use the principle of balance to solve some linear equations.

Example 2: Solve the equation $2x = 1$.
In order to solve this equation, we must find the value for x, which when put into the equation makes the equation true. That is, what number multiplied by 2 will equal 1? When we have found the answer to this question, we will have found the right value for x, and hence we have solved the equation.
Name of the Game: In order to solve an equation, we want to isolate the variable on one side of the equals sign, and all the numbers on the other side of the equals sign. When we have done this we get x = *some number*. That *some number* is the solution to the equation. How do we isolate the variable on one side of the equals sign and the numbers on the other side? By using the principle of balance! That is, by adding, subtracting, multiplying and dividing(by non-zero) numbers and variables on each side of the equation.

So for $2x = 1$, I want the x by itself on one side of the equation. I need to separate the x and the 2. Since 2 is not 0, I can divide both sides of the equation by 2. This will cancel out the 2 that is in front of the x. Then I will have the form x = *some number*, hence I will have a possible solution to the equation. Let's see how this is done:

Starting equation: $2x = 1$

Step 1: I want x isolated on one side of the equals sign, so divide by 2 on each side of equals sign

$$\frac{2x}{2} = \frac{1}{2}$$

Step 2: This can be rewritten as:

$$\frac{2}{2}x = \frac{1}{2} \quad \text{(Trick, see below)}$$

Step 3: But $\frac{2}{2} = 1$, so I have:

$$1x = \frac{1}{2} \quad \text{or} \quad x = \frac{1}{2}$$, which is the x = *some number* form.

Hence I have found a number for x. Is this the right value for x? Does it answer the question correctly, what times 2 equals 1? If the answer is yes, then we have solved the equation. Let's check and see:

Step 4: Check your answer: $2x = 1$ Putting in $\frac{1}{2}$ for x, I have

$$2\left(\frac{1}{2}\right) = \frac{2}{2} = 1$$ Which is true! Hence, I have solved the equation $2x = 1$. The solution is $x = \frac{1}{2}$.

There is a **trick** to this problem. In order to solve this problem you must recall a fact about fractions. It is that if you have a **product** (not the sum) of two numbers over a non-zero number, then you can split the expression up. This was done in step two above.

Remember:* If **a, **b** and **c** are numbers, **c** $\neq 0$, then $\frac{ab}{c} = \frac{a}{c}b = a\frac{b}{c}$.

Exercise 1: Solve the equation $3y = 5$. Write out all the steps in order to practice doing the steps.

Example 3: Solve the equation, $4x - 8 = x - 2$.

Step 1: Isolate all the x terms on one side of the equation by moving them all to one side or the other. This is done by adding or subtracting the x term to each side of the equation.

$$4x - 8 - x = x - 2 - x$$

Here I have subtracted the x term on the $x - 2$ (right hand side) from both sides.

Step 2: Collect the x terms together on both sides of the equation.

$$(4x - x) - 8 = (x - x) - 2$$

Step 3: Simplify the expression. Since variables are just place holders for numbers, we subtract variables like numbers. So $x - x = 0$, just like when you subtract like numbers.

$$3x - 8 = 0 - 2$$
$$3x - 8 = -2$$

Step 4: Now collect the **constants**, the numbers, on the other side of the equals sign. So here add 8 to both sides of the equals sign.

$$3x - 8 + 8 = -2 + 8$$

Step 5: Simplify the expression.

$$3x + 8 - 8 = 8 - 2$$
$$3x + (8 - 8) = 6$$
$$3x + 0 = 6$$
$$3x = 6$$ This is **not** the form $x = \textit{some number}$.

Step 6: Get the above into the $x = \textit{some number}$ form by dividing both sides by 3, which is not 0, so we can divide both sides by 3.

$$\frac{3x}{3} = \frac{6}{3}$$

Step 7: Rewrite and simplify.

$$\frac{3}{3}x = \frac{6}{3}$$
$$1x = 2 \quad \text{or} \quad x = 2$$

Step 8: Check the above number to make sure it is a solution. Putting $x = 2$ in the starting equation we have:

$$4(2) - 8 = 2 - 2$$
$$8 - 8 = 0$$
$$0 = 0$$ This is true!

Hence the solution to the equation $4x - 8 = x - 2$, is $x = 2$.

Observation: The $x = \textit{some number}$ form implies that the **coefficient**, the number in front of the variable x, must be one, and this means **positive 1**! In other words, suppose you solved an equation. You end up with $-x = 5$. This is not the solution! You must multiply both sides of the equation by -1 in order to get rid if the negative in front of the x. So what you would have is
$(-1)(-x) = (-1)(5)$ or $x = -5$ **not** 5.

Exercise 2: Solve the equation $3x = 2(x + 1) - (2 - x)$. First you have to multiply out the right hand side. Then proceed as in example 3. There is a special property that this equation has. You will discover that **before** you get to the $x = \textit{some number}$ form, you will find that all of the terms(the constants, as well as, the variables) cancel completely out. That is, you will get 0 = 0 before you find a value for x. When this happens for an equation, the equation is called an **identity**. This means that the solution set is all real numbers. No matter what number you put into the equation, that number is a solution. Try it. Put any number into the above equation before you try to solve it and see what happens.

Exercise 3: Solve the equation $x + 5 = x + 7$.
When you solve this you should get the statement 5 = 7. This is a false statement. When this happens the equation has no solution.

Example 4: Solve the following equation.

$$\frac{3x-4}{x} + \frac{1}{5} = 2$$

<u>Step 1</u>: First of all, since x is in the denominator of the first expression, you have to worry about **rule one** from the last lesson. Hence, x **can not be zero!** So 0 is excluded from the domain of x. Since the solution is in the domain of x, we know that 0 is not the solution.

<u>Step 2</u>: We need to get rid of the fractional expressions. This is done by multiplying each side by the **least common denominator**(commonly known as the LCD). Here the LCD is $5x$.

**Remember:* The LCD is the lowest expression that all the denominators of the fractions we are working with will divide evenly into.

So we are going to multiply each side of the equation by $5x$.

$$5x\left(\frac{3x-4}{x} + \frac{1}{5}\right) = (5x)(2)$$

<u>Step 3</u>: Now multiply each side out.

$$5x\left(\frac{3x-4}{x}\right) + 5x\left(\frac{1}{5}\right) = (5x)(2)$$

<u>Step 4</u>: Simplify each side.

$$\frac{(5x)(3x-4)}{x} + \frac{(5x)(1)}{5} = 10x$$

**Remember:* With fractions we cancel out like terms that appear in both the numerator and denominator (that is put the like terms over each other so we get 1). In the first fractional expression the x will cancel out and in the second fractional expression the 5 will cancel out as follows:

$$\left(\frac{x}{x}\right)(5)(3x-4) + \left(\frac{5}{5}\right)(x)(1) = 10x$$

$$(1)(5)(3x - 4) + (1)(x)(1) = 10x$$

<u>Step 5</u>: Multiply out the left hand side and simplify.

$$(5)(3x - 4) + x = 10x$$
$$15x - 20 + x = 10x$$
$$16x - 20 = 10x$$

Step 6: Now we proceed to finish solving the equation as in example 3. That is, isolate the variable x on one side of the equation and the constants on the other side of the equation.

$$16x - 16x - 20 = 10x - 16x$$
$$0 - 20 = -6x$$
$$-20 = -6x$$
$$\frac{-20}{6} = -x$$

Step 7: We are not done! We have to get rid of the negative sign in front of the x. So we multiply each side by -1.

$$(-1)\left(-\frac{20}{6}\right) = (-1)(-x)$$

$$\frac{20}{6} = x$$

**Remember:* We need to reduce $\frac{20}{6}$ to lowest form, that means cancel out all the common factors that 20 and 6 have. The largest factor they have in common is 2. This means, 2 can divide evenly into both 20 an 6 so we get:

$$\frac{20}{6} = \left(\frac{2 \bullet 10}{2 \bullet 3}\right) = \left(\frac{2}{2}\right)\left(\frac{10}{3}\right) = (1)\left(\frac{10}{3}\right) = \frac{10}{3}$$

So x is actually equal to $\frac{10}{3}$.

Step 8 : Check to see if the value we got for x is correct by putting it back into the original equation.

So putting $x = \frac{10}{3}$ into the equation we started with we have the following:

$$\frac{3\left(\frac{10}{3}\right) - 4}{\left(\frac{10}{3}\right)} + \frac{1}{5} = 2$$

Simplifying we get (I am not going through each step):

$$\frac{10-4}{\frac{10}{3}} + \frac{1}{5} = \frac{\frac{6}{1}}{\frac{10}{3}} + \frac{1}{5} = \frac{18}{3} + \frac{1}{5} = \frac{20}{10} = 2$$

Which is true! So the solution is $x = \frac{10}{3}$.

I would like to say one thing about the simplification above. **When you have a fraction over a fraction you multiply the outside numbers for the resulting numerator and the inside numbers for resulting denominator**. That is how I simplified the following:

$$\frac{\frac{6}{1}}{\frac{10}{3}} = \frac{6 \bullet 3}{10 \bullet 1} = \frac{18}{10}$$

The outside numbers here are 6 and 3, the inside numbers are 10 and 1. So the product of the outside numbers makes the resulting numerator 18 and the product of the inside numbers makes the resulting denominator 10.

Exercise 4: Solve the equation $\frac{2}{x+1} + 2 = 3$. What is the domain of x? What number is excluded from the domain of x? Why is that number excluded from the domain of x?

Looking Ahead: Being able to detect the numbers that are excluded from the variable's domain is very important. Sometimes, we will find solutions that are not in the variables's domain. When this happens, the equation may have no solution at all! Hence, not all equations are solvable.

Exercise 5: Solve the following equation:

$$\frac{2}{x+5} + \frac{1}{x+1} = \frac{-8}{(x+5)(x+1)} \quad .$$

First determine this equation's domain. Then cancel out the denominators.

Summary: Solving **linear equations** is possible because of the principle of balance. We add/subtract, multiply/divide (divide by non-zero terms) numbers and variables (the letters) to each side of a complicated equation in order to simplify the equation. The simplification process gives us an equation in the form $x =$ *some number*, which is **equivalent** to the original equation. By the principle of balance we know that this is the solution of the original equation. There were several tricks and properties of fractions that I went over. Please review them.

* *Remember:* In order to add/subtract fractions you have to put the fractions over a common denominator first.. For example, we want to add the following fractions:

$$\frac{2}{3} + \frac{4}{5}$$

You first need to find the **least common denominator** for all the fractions that will be added together. Here the least common denominator is the product of the denominators, 3 times 5, or 15. So we want to put 15 in the denominator of each fraction. **But**, if we do this we have to adjust the numerator also. Now, 15 divided by 3 is equal to 5, so we multiply the first numerator by 5. Since 15 divided by 5 is equal to 3, we multiply the second numerator by 3. By doing the above we now have the fractions over a common denominator and now all we do is add the numerators to get the sum of the fractions.

$$\frac{2}{3}+\frac{4}{5}=\frac{5\bullet 2}{15}+\frac{3\bullet 4}{15}=\frac{10}{15}+\frac{12}{15}=\frac{22}{15}$$

Notice that if you cancel out the common factors in $\frac{10}{15}=\frac{5\bullet 2}{5\bullet 3}=\frac{2}{3}$ you get the original fraction back.