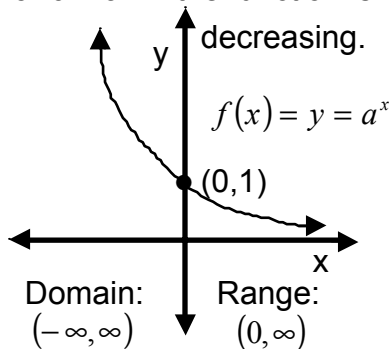


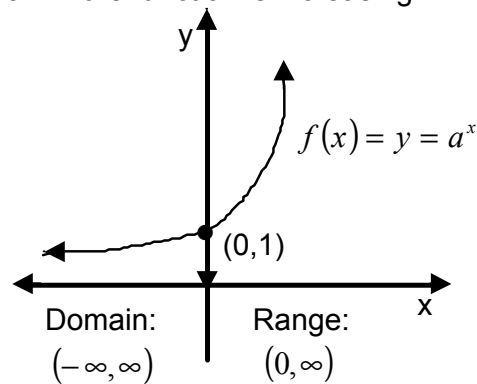
## Lesson 21: Exponential and Logarithmic Functions.

In this lesson we are going to learn about the *exponential* and *logarithmic functions*. These are two very important functions. An **exponential function** has the form  $f(x) = a^x$ , for  $a > 0$  and  $a \neq 1$ , this is the exponential function with **base  $a$** , and this is for all real  $x$ . The exponential expression  $a^x$  represents a unique positive real number. The domain of the exponential function is all real numbers  $x$ , and the range is all positive real numbers  $y$ . Also, since there is no value for which the function will be zero, we have a horizontal asymptote at the line  $y = 0$ , or the  $x$  – axis. The graph of the exponential function is as follows:

For  $0 < a < 1$  the function is decreasing.



For  $a > 1$  the function is increasing.



There were some properties of exponents that we looked at in lesson 1. We had for  $r, t, a$  and  $b$ , real numbers and  $a > 0$  and  $b > 0$  then:

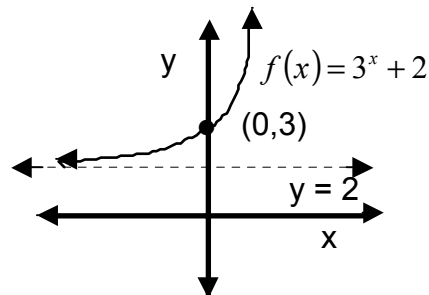
$$a^r \cdot a^t = a^{r+t}; \quad (a^r)^t = a^{r \cdot t}; \quad (ab)^r = a^r b^r; \quad 1^r = 1; \quad a^{-r} = \frac{1}{a^r} = \left(\frac{1}{a}\right)^r; \quad a^0 = 1$$

*One more property is : if  $a^r = a^t$ , then  $r = t$ .*

**Observation:** Since  $a^{-x} = \frac{1}{a^x} > 0$  there is **no such  $x$**  such that  $a^x = 0$ . Hence, the exponential function has no  $x$ -intercept, and so the  $x$ -axis is a horizontal asymptote that the function never crosses. This property is also why the range is all positive real numbers  $y$ .

**Example 1:** For the function  $f(x) = 3^x + 2$ , what kind of shift does this function have?

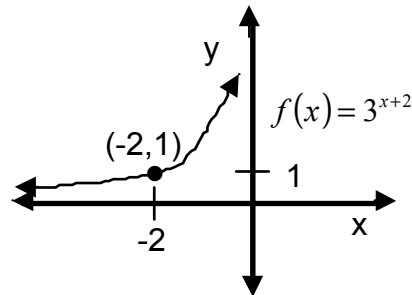
The center of the general exponential function occurs at the  $y$ -intercept or at the value  $x = 0$ . The center of the general exponential function is the point  $(0, 1)$ . So here, we have added 2 to the outside of the exponential function, the center is shifted by calculating:  $f(0) = 3^0 + 2 = 1 + 2 = 3$ . So the center is



shifted 2 units up from the general center of (0,1) or here the center shifts to (0,3). Also, the horizontal asymptote shifts up 2 units from  $y = 0$  to  $y = 2$ .

**Example 2:** What kind of shifting does the function  $f(x) = 3^{x+2}$  have?

Since we added 2 to the inside of the exponential function we shift the center 2 units to the left of the general center (0,1). So the new center is (-2, 1).



**Exercise 1:** What kind of shifting does the function  $f(x) = 2^{x-1} + 3$  have? Sketch the graph.

**Example 3:** Solve the exponential equation  $16 = a^{\frac{4}{3}}$ .

First, rewrite  $a^{\frac{4}{3}}$  as  $(\sqrt[3]{a})^4$ . Then we have the following:

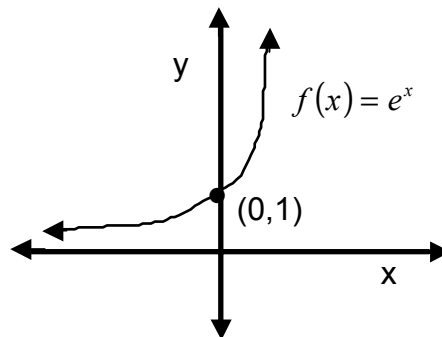
$16 = (\sqrt[3]{a})^4$ ;  $\pm\sqrt[4]{16} = \sqrt[3]{a}$ ;  $\pm 2 = \sqrt[3]{a}$ ;  $\pm 8 = a$ . The last step is obtained by cubing both sides. Checking both + 8 and - 8, we find that they are the correct solutions.

**Example 4:** Solve the exponential equation  $\left(\frac{1}{2}\right)^x = 8$ .

Using the above properties of the exponential we have the following:

$$\left(\frac{1}{2}\right)^x = 2^{-x} = 8 = 2^3. \text{ Hence, } -x = 3 \text{ or } x = -3.$$

There is a special base that occurs with exponential functions. The number  $e$  is an irrational number and appears often as the base to exponential functions. The value of  $e$  is approximately 2.71828... . The graph of the function  $f(x) = e^x$  is to the right. The domain is all real numbers  $x$ , and the range is all real positive real numbers  $y$ . The center or y-intercept is at the point (0, 1). Notice that the function is never equal to zero, and so the x-axis is a horizontal asymptote. Also, notice that the graph passes the horizontal line test, so the function is 1-1, which implies that it has an inverse function. (It turns out that its inverse is the logarithm function. We will learn about this function a little later.) Also notice that the function  $e^x$  is an increasing function.



Since  $e$  is approximately 2.718... > 1 then this function has the same properties as the function  $y = a^x$  when  $a > 1$

**Example 5:** Solve  $2xe^x - 6xe^x = 0$

Factor out the  $e^x$  part and then solve as follows:

$$(2x - 6x)e^x = 0 \Rightarrow (-4x)e^x = 0 \Rightarrow xe^x = 0 \Rightarrow x = 0 \text{ or } e^x = 0$$

However, the range of the exponential function excludes zero, so  $e^x \neq 0$ . Hence,  $x = 0$ .

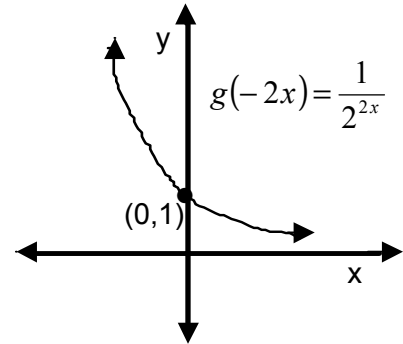
**Exercise 2:** Solve  $e^{x-2} \cdot e^{2x} = 1$ .

**Example 6:** State the domain, range, asymptotes, and graph the function  $f(x) = 1 + 2^{-2x}$ .

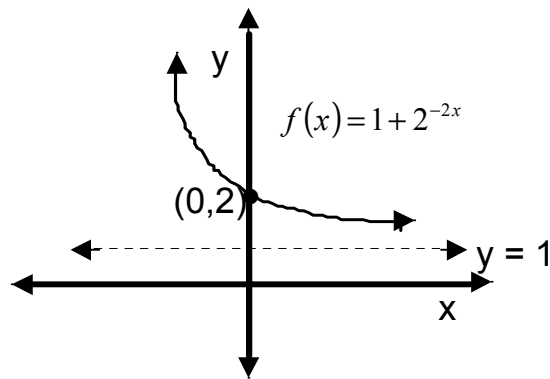
We are going to rewrite the  $f(x)$  as the sum of the constant function 1 and another function  $g(x)$ , which we compose with  $(-2x)$  so that we have the following:

$$g(x) = 2^x \text{ then } g(-2x) = 2^{-2x} \text{ and so } 1 + g(-2x) = 1 + 2^{-2x} = f(x).$$

Then we can easily see that the graph of  $g(-2x) = 2^{-2x} = \frac{1}{2^{2x}}$  is the following: the domain is all real numbers  $x$ , the range is all positive real numbers  $y$ , and the horizontal asymptote is the  $x$ -axis. The  $y$ -intercept for  $g(x)$  is the point  $(0, 1)$ .

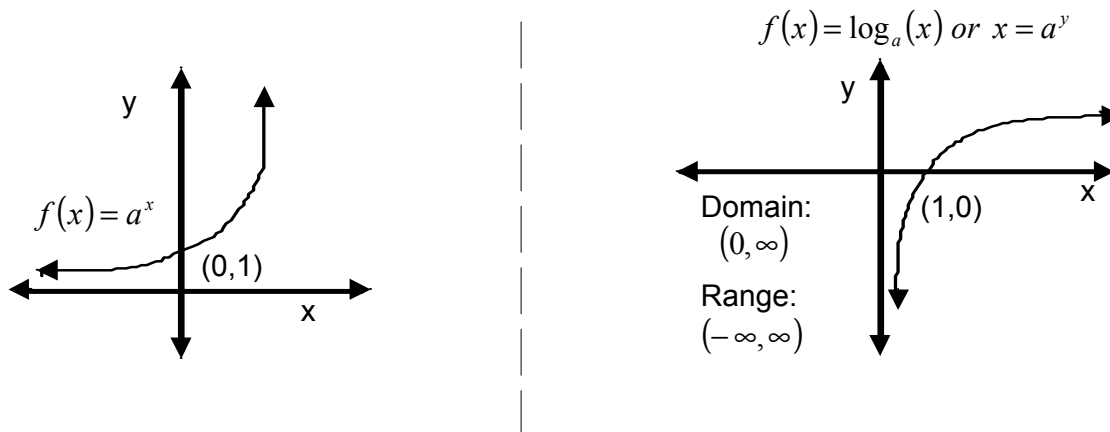


Now, since  $f(x)$  is just  $g(-2x)$  with a 1 added onto the outside of  $g(-2x)$ , all we need to do in order to get the graph of  $f(x)$  is shift the graph of  $g(-2x)$  up one unit to the point  $(0, 2)$ . This moves the horizontal asymptote up one unit to the line  $y = 1$ . This tells me that there is no value of  $x$  that I can choose that will give  $f(x)$  equal to one. Hence,  $f(x)$  gets close to one but never equals it. Because, there is no value of  $x$  that will make  $g(-2x) = 0$ . The domain of  $f(x)$  is all real numbers  $x$  and the range is all real positive numbers  $y > 1$ .



**Exercise 3:** State the domain, range, asymptotes, and graph the function  $f(x) = e^x + 2$ .

The inverse function of the exponential function is the **logarithm function**. So, if the exponential function is given by  $y = a^x$ , where  $a$  is the base, then the corresponding logarithmic function of base  $a$  is denoted as  $y = \log_a(x)$  where  $a > 0$ ,  $a \neq 1$ , and  $x$  is positive. The domain of the logarithm function is all positive real numbers, and the range is all real numbers, also, the  $y$ -axis is a vertical asymptote for the logarithm function. Because the  $y$ -axis is a vertical asymptote for the logarithm function, this means that you cannot take the logarithm of  $x = 0$ . However, this makes sense because, the  $x$ -axis is a horizontal asymptote for the exponential function so that when we take the inverse of it, the horizontal asymptote becomes a vertical asymptote for the logarithm function. Notice how the graphs are the reflection of each other through the line  $x = y$ :



**Observation:** For the equation  $y = \log_a(x)$  is equivalent to  $x = a^y$ .

**Example 7:** The  $\log_2(8) = 3$  is read as the logarithm base 2 of 8 is 3. Or as an exponential equation we have that  $y = 3$ , so this logarithm equation becomes  $2^3 = 8$ , or the base of 2 raised to the third power is 8.

**Observation:** To transform  $\log_2(8) = 3$  into its equivalent exponential form you can look at it as follows:

$$\log_2 8 \xrightarrow{\text{zag}} 3$$

$\xrightarrow{\text{zig}}$

We zig (step 1) over from the base of the log to what it is equal to then zag (step 2) back over to the expression inside the log function to get  $2^3 = 8$ .

**Example 8:** Find  $\log_3\left(\frac{1}{3}\right)$ .

We have that  $x = \frac{1}{3}$ ,  $a = 3$ , so we need to find the value of  $y$ . So we have the following;

$\log_3\left(\frac{1}{3}\right) = y$  so  $3^y = \frac{1}{3} = 3^{-1}$ . So  $y = -1$ . Hence,  $\log_3\left(\frac{1}{3}\right) = -1$ . Can you do this with the zig-zag technique?

**Exercise 4:** Express  $\log_a(1) = 0$  in exponential form. What property of the exponential is this?

Since the exponential and logarithm functions are inverses of each other we have that for  $f(x) = a^x$ ,  $f^{-1}(x) = \log_a(x)$  hence,  $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ . Hence, we have the following properties:

$\log_a(a^x) = x$  for  $x \in (-\infty, \infty)$ , and  $a^{\log_a(x)} = x$  for  $a > 0$ . What these properties tell us is that the same base cancels.

**Example 9:**  $\log_4(4^x) = x$  and  $9^{\log_9(x)} = x$ .

Recall that we had a special exponential base called  $e$ . We have a special logarithm that goes with the base of  $e$ . This is called the **natural logarithm**, and is written as  $\ln(x)$ . So we have that  $\ln(e^x) = x$  for  $x \in (-\infty, \infty)$  and  $e^{\ln(x)} = x$  for  $x > 0$ . Also,  $\ln(1) = 0$ , if and only if,  $e^0 = 1$ .

**Example 10:** Solve for  $x$  in  $\log_3(x) = 4$ .

Since the exponential function is the inverse of the logarithm function, the only way to get  $x$  out of the logarithm function is to raise both sides of the equation to exponents of the same base as the logarithm function. Here the base of the logarithm function is 3. So we have the following:

$$\log_3(x) = 4 \Rightarrow 3^{\log_3(x)} = 3^4; \text{ so } x = 3^4 = 81.$$

**Exercise 5:** Solve for  $x$  in  $\ln(x - 2) = 5$ . Hint; raise each side to the exponent of base  $e$ .

**Example 11:** Solve for  $x$  in  $e^{2-3x} = 2$ .

Here we will have to apply the log function with the same base as the exponent in order to get  $x$  out of the exponent. So since the base is  $e$ , we will apply the natural logarithm to each side of the equation. So we have the following:

$$\ln(e^{2-3x}) = \ln 2 \Rightarrow 2 - 3x = \ln 2 \Rightarrow -3x = \ln 2 - 2 \Rightarrow x = \frac{\ln 2 - 2}{-3} = \frac{2 - \ln 2}{3}.$$

**Exercise 6:** Solve for  $x$  in  $2^{x-5} = 16$  Hint; apply log to the base of 2 to each side, and remember

$$16 = 2^4.$$

**Summary:** We learned about two special functions in this lesson. We learned about the **exponential** function and the **logarithm** function. Since these functions are inverses of each other, the domain of the exponential function is the range of the logarithm function, which is all real numbers. The range of the exponential function is the domain of the logarithm function, which is all positive real numbers. For the exponential function there is no value of  $x$  that makes the function equal to 0, hence the x-axis is a horizontal asymptote. So that when we take the inverse of the exponential, or logarithm, we have that the y-axis is a vertical asymptote. Hence, we cannot take the logarithm of 0. Also, because these functions are inverses of each other, this property allows us to solve equations when  $x$  is contained inside the exponential or logarithm functions. In the next lesson we are going to learn some properties of the logarithm function that will allow us to solve more difficult equations involving exponential and logarithm functions.



Ahhh!