Lesson 1: Preliminaries and A Review of some Geometry and Complex Numbers.

Preliminaries:

Why is it when we have $2^{-3} = \frac{1}{2^3}$? We know that any number over itself is 1. For example, $\frac{2}{2} = 1$, this is true for any and all numbers (except zero, we'll get to this later). So we have the following;

$2^{-3} = 2^{-3}(1) = 2^{-3}\left(\frac{2^3}{2^3}\right) = \frac{2^{-3+3}}{2^3} = \frac{2^0}{2^3} = \frac{1}{2^3}$, I assume you know two things in order to calculate this. One is that any number to a power of zero is equal to 1, $2^0 = 1$. Also, that if a number is raised to two different powers and we take the product, we just add the powers together, $2^{-3} \bullet 2^3 = 2^{-3+3}$. Why is this last statement true? (I cannot tell you why the first statement is until we get to the lesson on exponentials.) Let□s look at an example: $2^3(2^2) = (2 \bullet 2 \bullet 2)(2 \bullet 2) = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 = 2^5 = 2^{3+2}$. Following this same line of thought we also have another fact: $(2^2)^3 = (2 \bullet 2)^3 = (2 \bullet 2)(2 \bullet 2)(2 \bullet 2) = 2^6 = 2^{2 \bullet 3}$. The three above examples represents three basic notions of exponentials that <u>you really need to remember</u>. The last two are similar, but note the differences!

Exercise 1: Can you show using the above information that: $\frac{2^2}{2^3} = 2^{2-3}$?

There is another exponential relationship that my students sometimes get confused. Suppose we have a fractional exponent. How do we calculate it? Here is an example, $4^{3/2} = \left(\sqrt{4}\right)^3 = 2^3 = 8$. Note how the numbers in the exponential are distributed.

Exercise 2: What does $x^{1/2}$ equal to?

Observation: Using Algebra we can write the above facts as general statements as follows: Let a, m, n be any real numbers, $a \neq 0$, then we have;

$$a^0 = 1;\ a^{-n} = \frac{1}{a^n};\ d^m \bullet d^n = d^{m+n};\ (a^m)^n = a^{m \bullet n};\ a^{m/n} = \left(\sqrt[n]{a}\right)^m\ a \geq 0;\ n \geq 2$$

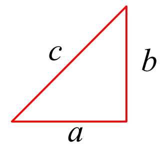
$a^1 = a$ (We don□t write the exponent when it is 1)

Exercise 3: How would you generalize the problem in exercise 1?

Review of some Geometry:

You will probably recognize the following formulas. I am not going to go through and derive them using geometry. The point here is to realize, that after reading the introduction, it is Algebra that allows us write generalized formulas for things using letters. So each time we need to calculate the area, etc., all we need to do is plug in the numbers of our particular case into the correct general formula.

Lets first talk about triangles. We have all heard of **right triangles.** These are triangles with one angle of 90 degrees. The most well known fact about right triangles is the **Pythagorean Theorem.** Simply stated it says that the square of the hypotenuse is equal to the sum of the squares of the legs. This is the infamous equation, $a^2 + b^2 = c^2$, where a is the length of one leg, b is the length of the other leg, and c is the length of the hypotenuse.



The **Converse of the Pythagorean Theorem** states simply that, if you have a triangle in which the square of one side is equal to the sum of the squares of the other two sides, then the triangle has a 90 degree angle opposite the longest side, and hence the triangle is a right triangle.

Looking Ahead: In order to derive the Pythagorean Theorem, one must know how to *complete the square.* Most of my students do not know how to do this. It seems to be a fact lost in high school mathematics these days. We will go over this in Lesson 6. If you plan to go into more advanced math courses you **must know how to complete the square!**

Two triangles are **congruent**, if we can lay one exactly on top of the other without changing its shape. If two triangles are congruent then the length of the sides are equal, as well as, the measure of the angles.

Some other geometric formulas are:

Triangles: For a Triangle with base b, and height h: Area = $\frac{1}{2}bh$.

Rectangles: For a Rectangle with width w, and length l: Area = lw; Perimeter = $2l + 2w$.

Rectangular Box: For a Rectangular Box with length l, width w, and height h: Volume = lwh.

Circle: For a Circle with radius r,
Diameter = $2r$; Area = πr^2; Circumference = $2\pi r$

Advanced Observation: Area and Perimeter are what are called two-dimensional measurements, because they depend on two measurements. Volume is called a three dimensional measurement, because volume depends on three measurements; length, width and height.

Looking Ahead: We will come back to the dimensional observation when we get to Lesson 13. In that lesson we will learn about functions. Area is considered to be a two-dimensional function because it depends on two variables. Volume is considered to be a three-dimensional function because it depends on three variables. A <u>variable</u> is the letter we use in place of a specific number in the formulas.

<u>Complex Numbers:</u>

Many of my students have seen or heard a little about Complex Numbers. All the numbers we use daily are what are called **Real numbers**. Complex numbers are used when real numbers cannot be used. For example, **in real numbers you cannot have a negative square root**. That is, in real numbers $\sqrt{-2}$ does not exist, but in the world of complex numbers it does. We will find out why this is the case in Lesson 14. Although much of your work in this course and other courses are with real numbers, you do need to know a few things about complex numbers.

A complex number has the form $a + bi$, where the a and b are real numbers. We say that a is the <u>real part</u>, and b is the <u>imaginary part</u>. Notice that the i goes with the b part. When we have the $a + bi$ form, this is called <u>standard form</u>. If $a = 0$, so that all we have is the form bi, we call this a <u>pure imaginary number</u> .

So what exactly is i? Here is what i is good for, $i = \sqrt{-1}$. *This is what allows negative square roots to exist in Complex Numbers***.** In other words, $(i)^2 = -1$.

Looking Ahead: The fact that $(i)^2 = -1$ is important because it allows us to solve the equation $x^2 = -1$. In real numbers, this equation is **not** solvable. However, sometimes we end up with a negative square root when we factor polynomials. We will see what this means in Lesson 20. We will also be solving quadratic equations in Lesson 6, so we will see how to solve for x in the equation $x^2 = -1$.

The powers of i follow a pattern:

$(i)^1 = i$; $(i)^2 = \left(\sqrt{-1}\right)^2 = -1$; $(i)^3 = i \bullet (i)^2 = i \bullet (-1) = -i$; $(i)^4 = (i)^2 \bullet (i)^2 = (-1) \bullet (-1) = 1$;
$(i)^5 = (i)^4 \bullet i = 1 \bullet i = i$

**Remember:* A negative number times a negative number equals a positive number. In the above example we have (- 1)(-1) = 1. This leads to the fact that a negative number squared is positive. Such as, $(-2)^2 = (-2)(-2) = 4$.

Exercise 4: Following the above examples write out what, $(i)^6$, $(i)^7$, $(i)^8$, are equal to**.** Be careful, the last one is tricky! Do you see the pattern?

Now, complex numbers can be added, subtracted, multiplied and divided. However, the way these operations are done can be a little tricky. Let $a + bi$ and $c + di$ be two complex numbers, <u>**where *a, b, c, d* are any real numbers**</u> (this includes fractions too).
Then we have the following:

<u>Equality property</u>: $a + bi = c + di$ only when $a = c$ and $b = d$.
<u>Sum</u>: $(a + bi) + (c + di) = (a + c) + (b + d)i$
<u>Difference</u>: $(a + bi) - (c + di) = (a + bi) + (-c - di) = (a - c) + (bi - di)$
<u>Product</u>: $(a + bi)(c + di) = a(c + di) + bi(c + di) = ac + adi + cbi + bd(i)^2$
$= ac + adi + cbi - bd = (ac - bd) + (ad + cb)i$

Notice how we group the real parts together and the imaginary parts together. Don't worry if you don't understand each step. Some steps are techniques used in solving equations. I just wanted you to see the steps all worked out, you can come back later and look the steps over again.

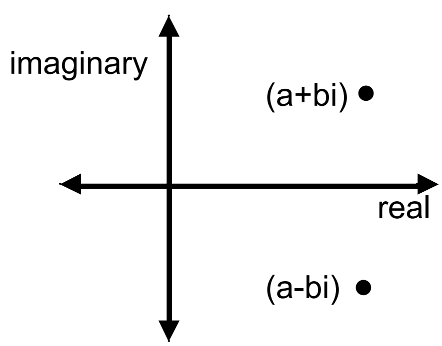
Note: When we put parentheses around math expressions as in the product rule above, this means to multiply the terms out as I did in the above rule. Also, (2)(3) = 2 • 3 = 6.

Example 1: Find the product;

$(2+4i)(3+6i)$ Here a = 2, b = 4, c = 3, d = 6. You do not have to multiply it all out if you don't want to. Just apply the above product formula as follows:
$(2 + 4i)(3 + 6i) = (ac - bd) + (ad + cb)i = (2 \bullet 3 - 4 \bullet 6) + (2 \bullet 6 + 3 \bullet 4)i = (6 - 24) + (12 + 12)i$
$= -18 + 24i$.

**Remember:* First you multiply numbers then add/subtract them. That is, multiplication/division is done before addition/subtraction.

The quotient gets a little sticky, so we will do it last. However, complex numbers have another property that I'd like to show you. Complex numbers have a **conjugate.** This means that if we have a complex number $a + bi$ (this is sometimes represented as one letter, z), then its conjugate is denoted by $\overline{a+bi}$, and $\overline{a+bi} = a - bi$. The conjugate is the reflection of the point $a + bi$ through the real axis. See picture below:



Exercise 5: Calculate using the above properties $(a+bi)\overline{(a+bi)}$. First use the definition of the conjugate, then multiply the terms out. What you should end up with is $a^2 + b^2$, which is a nonnegative real number! *Moral*, any complex number multiplied by its conjugate is a positive real number.

There is one more thing about conjugates. If you take the conjugate of the conjugate, you get the complex number back, $\overline{\overline{(a+bi)}} = a + bi$.

Now, on to <u>division</u>. Let $a + bi$ and $c + di$ be two complex numbers. Then we have the following quotient property:

$$\frac{a+bi}{c+di}=\left(\frac{a+bi}{c+di}\right)\bullet 1=\frac{a+bi}{c+di}\bullet\frac{c-di}{c-di}=\frac{ac+bd}{c^2+d^2}+\left(\frac{bc-ad}{c^2-d^2}\right)i\ ;\ \text{where } c^2+d^2\neq 0\,.$$

Exercise 6: Fill in the missing steps in the above calculation.

Observation: First of all, if we have any nonzero complex number over itself that is equal to one, as in the above $\frac{c-di}{c-di}=1$. Also, we multiplied $\frac{a+bi}{c+di}$ by $\frac{c-di}{c-di}$ because *c* - *di* is the complex conjugate of *c* + *di* that is in the denominator. Their product gives us a real number in the denominator, which is what we want.

Example 2: Find the quotient;

$$\frac{5-i}{3+i}=\left(\frac{5-i}{3+i}\right)\left(\frac{3-i}{3-i}\right)=\frac{15-5i-3i-1}{9+1}=\frac{14}{10}-\frac{8}{10}i=\frac{7}{5}-\frac{4}{5}i$$

Advanced Observation: Any **real** number can be written as a complex number. This is done simply by putting a zero in front of the *i* in the imaginary part. For example, 2 = 2 + 0*i*. So for any real number *a, a* = *a* + 0*i*. Because of this fact, the real numbers are a **subset** of the complex numbers. This means that the space of real numbers is contained inside the space of complex numbers. This just tells us that the space of complex numbers is bigger then the space of real numbers. This is important in advanced math and engineering. Sometimes the space of real numbers is projected into the space of complex numbers. This is done because some calculations that are not possible in the real space can be done in the complex space.

Summary: I know this is a lot to take in, but don't get over whelmed or give up. This is just preliminary stuff. The stuff on exponentials is the most important thing right now. The complex number stuff is not really needed until Lesson 20. You can come back and review complex numbers then. The Geometry equations are there basically as reference and review. Let's proceed to the heart of Algebra, solving equations! That will be fun.