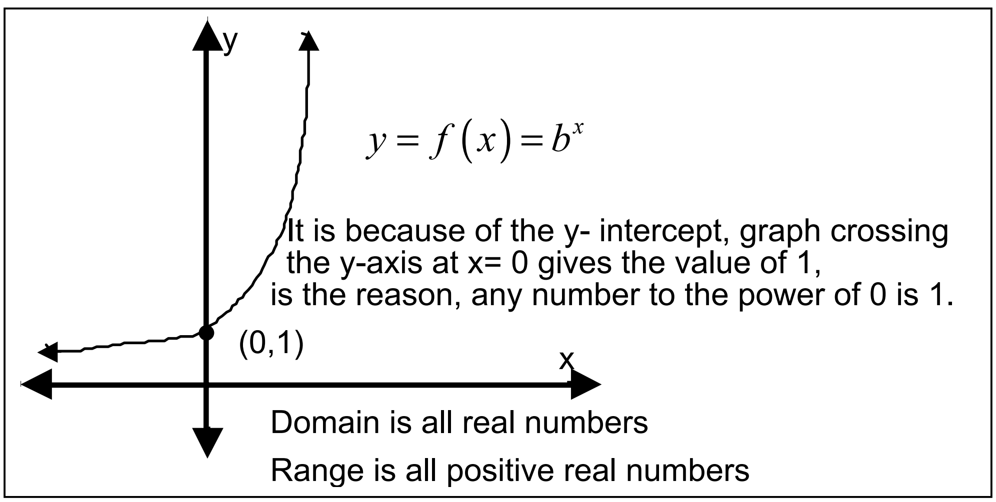
Chapter 7: Exponential and Logarithmic Functions.

Section 7.0A: Review of Exponentials and Logarithms.

A function in which the variable *x* appears in the exponent is called an **exponential function**. For example, $y = 2^x$ is an exponential function, where *y* is the **dependent** variable and *x* is the **independent** variable.

> **Exponential Function**
>
> An equation of the form $y = b^x$ where b is a positive constant and not equal to 1, is called an exponential function and *b* is called the base.

The graph of an exponential function looks like:



Recall that an **irrational number** is a number whose decimal representation never ends or repeats, like the number π. Another irrational number that appears often in mathematics is $e = 2.71828182\ldots$. This is an irrational number and when it appears as a base we use the letter *e* instead of its decimal notation. So, when we see $f(x) = e^x$ this is really just $f(x) = 2.71828182...^x$. We use a calculator to calculate e^x.

Most calculators have an e^x button. The graph of this function looks like the picture above. The exponential function grows very rapidly when x is large. This is used extensively in population growth models.

Now let us look at the question, ten raised to what power gives 100? 346? The idea of finding the exponent to which a number must be raised in order to get some particular number is the central concept of a **logarithm**. Hence " 3 raised to what power gives 9?" can be rephrased as "What is the logarithm (base 3) of 9?" The second question is written mathematically as $\log_3 9 = x$ which is written in terms of exponentials as $3^x = 9$.

Logarithm Definition

$\log_b u = v$ or the logarithm of u is the same as $b^v = u$.
In either case, b is the base and must be positive.

It is easy to convert from the logarithm to exponential as follows:

$$\log_b u \underset{1,\, zig}{\overset{2,\, zag}{\rightleftarrows}} v \quad \Rightarrow \quad b^v = u$$

Example 1: Find the $\log_2 8 = v$. Written in terms of exponentials we have

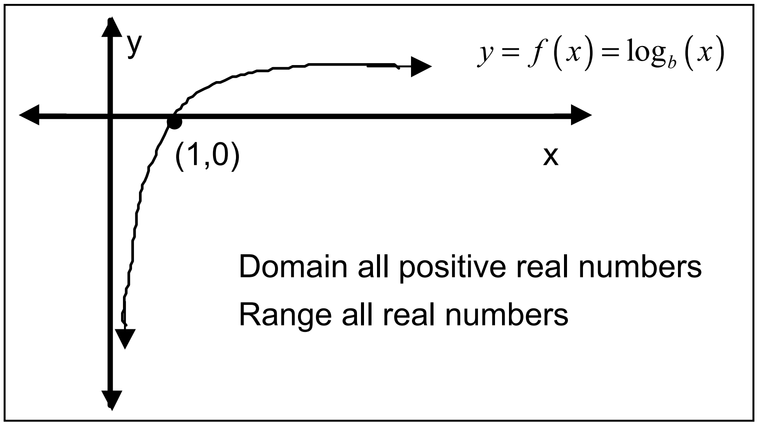
$2^v = 8 \Rightarrow v = 3$.

A **common logarithm** is a logarithm to the base of 10.

Common Logarithm

$y = \log x$, or $\log_{10} x = y$, is the same as $10^y = x$.

Again, we can use a calculator to find the logarithm of a number. The graph of the logarithm function is the inverse of the exponential function. Hence, the logarithm of **negative numbers does not exist**! The graph of the logarithmic function is below:



Recall that we had a special base called *e*. For this base we have as its associated logarithm the **natural logarithm** or ln.

Natural Logarithm definition:

$y = \ln x$ (or $\log_e x = y$) means the same as $e^x = y$.

Again we can use a calculator to evaluate the ln of a number. In fact, a calculator can only evaluate logarithms to the base of 10 or ln. If the base is different, we have to use the change of base formula to evaluate the logarithm. So we use the change of base to put logarithms of different bases into base 10 logarithms. The change of base formula is as follows:

If **a** > 0, **b** > 0 and **a** $\neq 1$, **b** a $\neq 1$ **a**nd x is a positive real number then

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

<u>Section 7.0B</u>: Review of Properties of Logarithms.

Recall that the logarithm and exponential functions are inverses of each other. That is, like bases cancel. For example, $\log_2 2^5 = 5$. Hence in general, the logarithm function undoes the exponential and the exponential function undoes the logarithm function, when they are both to the same base.

Inverse Properties

$$\log_b\left(b^v\right) = v \text{ and } b^{\log_b v} = v$$

$$\ln\left(e^v\right) = v \text{ and } e^{\ln v} = v$$

<u>Example 1</u>: Simplify $5^{\log_5(2+x)}$ and $\ln\left(e^{\left(x^2-3\right)}\right)$.

Solution: Since in both cases the bases are the same, the functions cancel each other out, that is:

$$5^{\log_5(2+x)} = 2 + x \text{ and } \ln\left(e^{\left(x^2-3\right)}\right) = x^2 - 3$$

An **exponential equation** is an equation in which *x* appears in the exponent. Hence, in order to solve an exponential equation, we use the inverse property of the exponential and logarithmic functions.

<u>Example 2</u>: Solve $e^{x+2} = 1$.

Solution: Since the inverse of *e* is ln, we take the ln of both sides to get:

$$\ln\left(e^{x+2}\right) = \ln 1 \Rightarrow x + 2 = 0 \Rightarrow x = -2.$$

Steps for solving Exponential Equations:

1.) Isolate the exponential, $e^A = B$.
2.) Take the natural logarithm of each side.
3.) Use the inverse property to simplify.
4.) Solve.
5.) Use your calculator to check the answer.

These steps apply to any exponential base, but you must take the appropriate logarithm base of each side of the equation.

There are several rules to the logarithmic function that we need to restate. These properties can be found by just expressing the properties of the exponential into logarithmic form. The properties are as follows:

for $x > 0$, $y > 0$ and **r** > 0 and **a** > 0 , **a** $\neq 1$ we have

1.) $\log_a(xy) = \log_a(x) + \log_a(y)$ (2.) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

3.) $\log_a(1) = 0 \text{ and } \log_a(a) = 1$. In general, $\log_a(a^r) = r$

4.) $\log_a(x^r) = r\log_a(x)$.

5.) $\log_a(x) = \log_a(y)$, if and only if, $x = y$.

We can use these properties to express logarithm equations into different forms.

Example 3: Solve $1.03^x = 2$.

Solution: We can take the log base 10 of each side and then use property 4 above to solve for x. Hence, we have

$$\log(1.03^x) = \log(2) \Rightarrow x\log(1.03) = \log(2) \Rightarrow x = \frac{\log(2)}{\log(1.03)} \approx 23.45$$

Chemists define pH by the formula pH = $-\log\left[H^+\right]$, where $\left[H^+\right]$ is the hydrogen ion concentration measured in moles per liter. A pH of 7 is considered neutral, below that is a base and above that is acidic.

Steps for solving Logarithmic Equations

1.) Get all log terms on one side of the equation.
2.) Combine the log terms into one log term using the properties.
3.) Exponentiate each side.
4.) Use the inverse property and simplify.
5.) Solve.

Example 4: Solve the equation log(3*x*) - log(1.3) = 2.4.

Solution: Using property 2 above, and the inverse property of base 10, we have:

$$\log(3x) - \log 1.3 = \log\left(\frac{3x}{1.3}\right) = 2.4$$

$$\Rightarrow 10^{\left(\log\left(\frac{3x}{1.3}\right)\right)} = 10^{2.4} \Rightarrow \frac{3x}{1.3} = 10^{2.4} \Rightarrow 3x = 10^{2.4}(1.3) \Rightarrow x \approx 108.85$$

Example 5: Fresh-brewed coffee has a hydrogen ion concentration of about 1.3×10^{-5} moles per liter. Determine the pH of fresh-brewed coffee.

Solution: We know that pH = $-\log\left[H^+\right]$, hence we have

$$\text{pH} = -\log\left[H^+\right] = -\log\left(1.3\times10^{-5}\right) = -(-4.88) \approx 4.9.$$

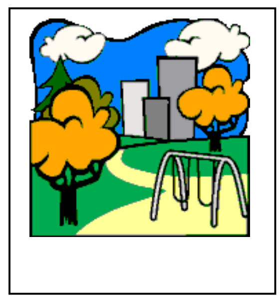
Section 7.1: Exponential Growth.

We will now look at how the exponential function is used in measuring growth, such as population growth or inflation. First we need to define the **delta** (Δ) notation. This is used as shorthand to describe the "change in" a quantity. A **rate of change** always means one change divided by another change.

<u>Example 1</u>: Two years ago, Anytown, U.S.A., had a population of 30,000. Last year, there were 900 births and 300 deaths, for a net growth of 600. There is no immigration to or emigration from the town, so births and deaths were the only sources of population change. The average growth rate was therefore $\frac{\Delta p}{\Delta t} = \frac{600 \text{ people}}{1 \text{ year}}$. What is the most likely average growth rate for Anytown this year, assuming no change in living conditions?

Solution: During the first year the population growth was the growth rate divided by the total population or $\frac{\Delta p / \Delta t}{p} = \frac{600 \cancel{\text{ people}} / 1 \text{ year}}{30{,}000 \cancel{\text{ people}}} = 2\% \text{ per year}$. Hence, in the following year there should again be a 2% growth in the population but now p = 30,600. So we have following : $\frac{\Delta p / \Delta t}{30{,}600} = 2\% \Rightarrow \Delta p / \Delta t = 2\% \text{ of } 30{,}600 = 612$. Hence, we can predict that the average growth rate for Anytown this year would be 612 people/year.

In the above example the average growth rate was $\frac{\Delta p}{\Delta t}$ was a constant percent of the current population. Inflation to is a constant percent of a current value. We are looking at situations where the average growth rate $\frac{\Delta y}{\Delta t}$ is a constant percent of the current value of y, that is, in which $\frac{\Delta y}{\Delta t} = k \bullet y$, where k is some constant. Using calculus we determine that if a quantity y behaves in such a way that the average growth rate $\frac{\Delta y}{\Delta t}$ is a constant percent of the current value of y, then the size of y at some later time can be predicted using the equation $y = ae^{bt}$ where t is time and a and b are constants. A **mathematical model** is an equation that describes a real world subject. The exponential equation $y = ae^{bt}$ is called the **exponential model** because it is an exponential equation and is used to describe subjects such as populations and real estate appreciation. This model allows us to predict the value of y and its growth rate with just a few data. Because many different quantities have a growth rate proportional to their size, the applications of this model are wide and varied.



Example 2: In the year 2000, Anytown, U.S.A. had a population of 30,000. in the following year, there were 900 births and 300 deaths, for a net growth of 600. There was no immigration to or emigration from the town, so births and deaths were the only sources of population change.

a.) Develop the model that represents Anytown's population.

b.) Predict Anytown's population in the year 2006.

Solution: To develop the model, we can use $y = ae^{bt}$ since the population growth rate is constant percent of the size of the population. We need to find a and b for our model. Let 2000 be the beginning of time, so let 2000 be t = 0; then 2001 is t = 1 and 2006 is t = 6. If we count the population in thousands, we have the ordered pairs (0,30) and (1,30.6). Now substituting the first ordered pair in the equation we have:

(t,p) = (0,30) so, $y = ae^{bt} \Rightarrow 30 = ae^{b \bullet 0} = ae^{0} \Rightarrow a = 30$. Hence, our initial population is the value of a. Now substitute the second ordered pair (1,30.6) in to the model.

$$y = ae^{bt} \Rightarrow 30.6 = 30e^{b \bullet 1} \Rightarrow$$

$$\frac{30.6}{30} = e^{b} \Rightarrow 1.02 = e^{b} \Rightarrow \ln 1.02 = \ln\left(e^{b}\right) \Rightarrow b \approx .0198026273$$

Thus our model for population growth in Anytown is $p = 30e^{.0198026273t}$.

Hence in the year 2006, we predict the population at time t = 6 to be:

$p = 30e^{.0198026273(6)} \approx 33.784$ so probably about 33,784.

Notice in the above example that b is close to the population growth of 2%. In general, b is always close to the relative growth rate, and the smaller the numbers are the closer they are.

Example 3: A house is purchased for $140,000 in January of 2000. A year later, the house next door is sold for $149,800. The two houses are of the same style and size and are in similar condition, so they should have equal value.

a.) Develop the mathematical model that represents the home's value.

b.) Find when the house would be worth $200,000, assuming the rate of appreciation for houses continues unchanged.

Solution: Let t be time and v be the value of the house. Then the ordered pairs, (t,v), are (0,140) and (1,149.8) where the value is in thousands of dollars. So substitute the first ordered pair into the equation $y = ae^{bt}$ to get the initial conditions;

$140 = ae^{b \bullet 0} \Rightarrow a = 140$. Now substitute the second ordered pair into the equation

$$149.8 = 140e^{b \bullet 1} \Rightarrow \frac{149.8}{140} = e^{b} \Rightarrow \ln 1.07 = \ln e^{b} \Rightarrow b \approx .067658$$

Hence our model is $v = 140e^{.067658t}$. Now to figure out when the house will be worth $200,000, substitute this value for v and solve for time t:

$$200 = 140e^{.067658t} \Rightarrow \ln \frac{200}{140} = \ln e^{.067658t} \Rightarrow t = \frac{\ln \frac{200}{140}}{.067658} \approx 5.27 \text{ years.}$$

Remember that this calculation is only a prediction and will be accurate if the appreciation continues at 7%.

Steps in developing an Exponential Model

1.) Write the given information into two ordered pairs (t,y).
2.) Substitute the first ordered pair into the model $y = ae^{bt}$ and simplify to get the value of a.
3.) Substitute the second ordered pair into the model and simplify to get the value of b.

Recall the section on compound interest, could there be a relationship between the inflation of real estate and compound interest? In example 3 above, we found that if a $140,000 house increased in value by 7% from January 2000 to January 2001, where 7% is the house's relative growth rate, is this the same as depositing $140,000 in an account for a rate of 7% compounded annually?

Example 4: In January 2000, $140,000 is deposited in an account that earns 7% interest compounded annually.

a.) Find the future value in January 20001.

b.) Find when the account would hold $200,000.

c.) Develop a compound interest model that can be used to answer questions involving the future value of the account.

Solution: a.) Recall the compound interest formula, then fill in the amounts and simplify: $FV = P(1+i)^n = 140,000(1+.07)^1 = \$149,000$.

b.)Now we are asked to find out when the account will be worth $200,000, so we will again use the compound interest formula, then solve for the exponential like we have done so far in this section.

$$FV = P(1+i)^n \Rightarrow 200,000 = 140,000(1+.07)^n \Rightarrow \frac{200,000}{140,000} = (1+.07)^n \Rightarrow \frac{20}{14} = (1+.07)^n$$

$$\ln\left(\frac{20}{14}\right) = \ln(1.07)^n \Rightarrow \ln\left(\frac{20}{14}\right) = n\ln(1.07)$$

$$\Rightarrow \frac{\ln\left(\frac{20}{14}\right)}{\ln(1.07)} = n \Rightarrow n \approx 5.27...$$

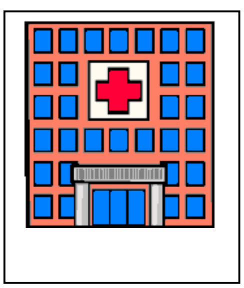
Since n is the number of compounding periods and interest is compounded annually, n = 5.27 means that it will take 5.27 years for the account to hold $200,000. Although this is mathematically correct, if the interest is compounded annually the .27 does not make sense. After 5 years the account will hold approximately $196,357.24, since interest is compounded annually the .27 does not help the amount of the account, but after 6 years there will be approximately $210,102 after 6 years, the will hold more than $200,000.

c.)To develop a compound interest model that can be used to answer questions involving the future value of the account, substitute $140,000 for P and 7% = .07 for i into the compound interest formula.

$FV = P(1+i)^n \Rightarrow FV = 140,000(1.07)^n$ in this model, n must be a whole number of years.

<u>Section 7.2</u>: Exponential Decay.

The exponential function is not only used for growth, but also for decay. When a scientist states that an artifact is about 10,000 years old, they are basing that on the radioactive decay of certain chemical elements. A radioactive substance is not stable; over time it transforms into a different substance. This is called **radioactive decay**. A larger amount of a radioactive material experiences more decay. That is, the rate of decay is proportional to the amount of radioactive substance present. Hence, we can use the exponential model $y = ae^{bt}$ that we developed earlier. However, since we are talking about the quantity left, we use Q instead of y, hence we have $Q = ae^{bt}$.



<u>Example 1</u>: Hospitals use the radioactive substance iodine-131 in research. It is effective in locating brain tumors and in measuring heart, liver and thyroid activity. A hospital purchased 20 grams of the substance. Eight days later, when a doctor wanted to use some of the iodine-131, he observed that only 10 grams were left (the rest had decayed).

a.) Develop a mathematical model that represents the amount of iodine-131 present.

b.) Predict the amount remaining two weeks after purchase.

Solution: We develop the model as we did previously. We use the two ordered pairs (t,Q), where originally we have (0,20). Eight days later we had 10 grams, (8,10).

$Q = ae^{bt} \Rightarrow 20 = ae^{b0} \Rightarrow 20 = a \bullet 1$; hence, $a = 20$. Now we need to solve for b using the second ordered pair (8,10) as follows:

$$Q = ae^{bt} \Rightarrow 10 = 20e^{b\bullet 8} \Rightarrow \frac{10}{20} = e^{8b} \Rightarrow \ln(.5) = \ln\left(e^{8b}\right) \Rightarrow \ln(.5) = 8b$$

$$\Rightarrow b = \frac{\ln(.5)}{8} \Rightarrow b \approx -0.086643397$$

Hence our model is $Q = 20e^{-0.086643397t}$ which tells me the quantity of iodine-131 remaining after t days after purchase. So, after 2 weeks or 14 days, the quantity left is

$$Q = 20e^{-0.086643397(14)} = 5.946035575\ldots \approx 5.9 \text{ grams remain.}$$

Notice in the above example that b is negative, this means that in the equation, the quantity is decreasing. Where as before, b was positive which meant that the amount was increasing.

Exponential Growth or Exponential Decay

In general, any quantity for which the rate of change is proportional to the amount present can be modeled by the formula $y = ae^{bt}$;

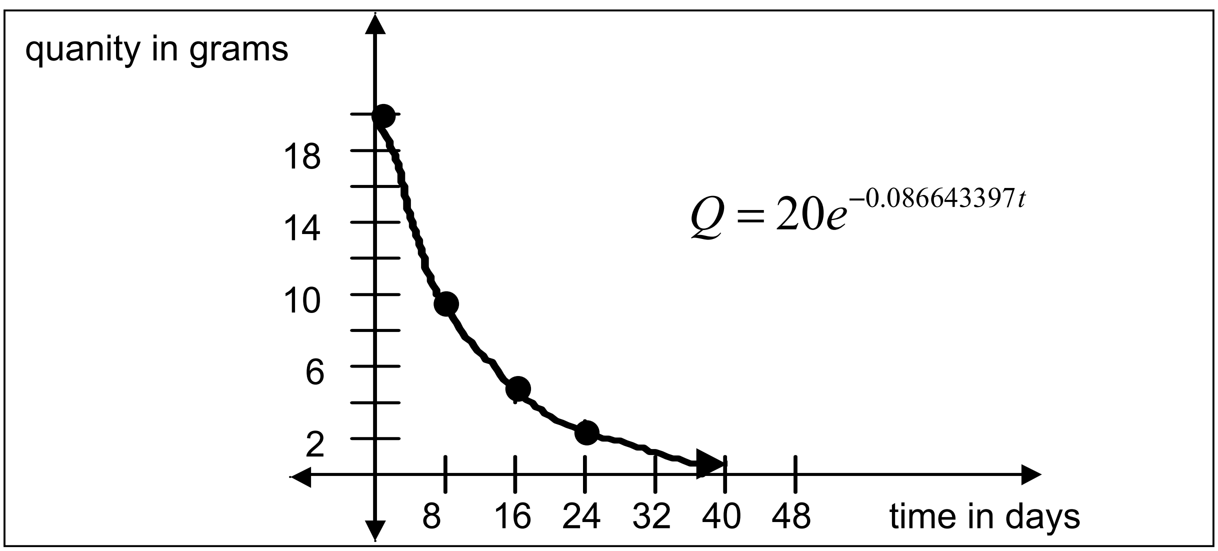
1.) If $b > 0$, then y is growing exponentially.
2.) If $b < 0$, then y is decaying exponentially.

A radioactive substance is not stable. One way to measure its instability is to determine the **half-life** of the substance. In the example above, we found that after 8 days, 20 grams of iodine-131 decayed to 10 grams. Hence the half-life of iodine-131 is 8 days. This means that in 8 more days, the 10 grams will decay to 5 grams.

Example 2: Use the model $Q = 20e^{-0.086643397t}$ to calculate the amount of iodine-131 remaining after 16 days of purchase and 24 days after purchase.

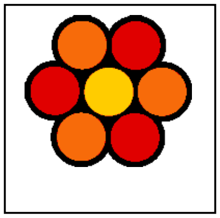
Solution: So after 16 days we have $Q = 20e^{-0.086643397(16)} = 5$ and after 24 days we have $Q = 20e^{-0.086643397(24)} = 2.5$, notice how both of these amounts of time is adding half-life periods, so the quantity of iodine-131 is decreasing by halves.

Notice that in example 1 and example 2 we found the following ordered pairs, (0,20), (8,10), (16,5), (24,2.4) where each ordered pair is (time, quantity left). So drawing a coordinate plane where the **x-axis is time**, and the **y-axis is quantity of iodine-131** left, which is dependent on the amount of time that has passed, thus it is the dependent variable and time is the independent variable. Plotting these points we get the following graph of the exponential decay of iodine-131.



Using the model $Q = ae^{bt}$, we can compute the **average decay rate** $\frac{\Delta Q}{\Delta t}$.

We can also compute the **relative decay rate** by calculating $\left(\frac{\Delta Q}{\Delta t}\right) \Big/ Q$. Since Q is decreasing, ΔQ will be negative, as will the relative decay rate.



Example 3: Plutonium-239 is a waste product of nuclear reactors. How long will it take for this waste to lose 99.9% of its radioactivity and therefore be considered relatively harmless to the biosphere? The half-life of plutonium-239 is 24,400 years.

Solution: Regardless of the initial amount of plutonium-239, we need to determine the time required for .1% of the radioactivity to remain. We first need to determine the model for plutonium-239, using $Q = ae^{bt}$ if a = the initial amount of plutonium-239, we have for the initial conditions of $(0,a)$ and for its half-life $(24{,}400, \frac{a}{2})$. Since we don't have an initial value for a, we will not change the basic model if we substitute the first ordered pair into it. If we substitute the second ordered pair into the model and simplify, we get:

$$Q = ae^{bt} \Rightarrow \frac{a}{2} = ae^{b \bullet 24{,}400} \Rightarrow \frac{1}{2} = e^{24{,}400b} \Rightarrow \ln(.5) = \ln\left(e^{24{,}400b}\right)$$

$$\ln(.5) = 24{,}400b \Rightarrow b = \frac{\ln(.5)}{24{,}400} \approx -.000028407$$

We still need to find the time t for which .1% of a remains, this means we need to take Q to be $.001a$. Putting this into the model, we have;

$$.001a = ae^{-.000028407t}$$

$$\Rightarrow .001 = e^{-.000028407t} \Rightarrow t = \frac{\ln(.001)}{-.000028407} \approx 240,000 \text{ years.}$$

Radioactive substances are used to determine the age of fossils and artifacts. The procedure is based on the fact that two types of carbon occur naturally. Carbon-12 is stable and carbon-14 is radioactive. The amount of carbon-14 is relatively small, only one atom for every 1 trillion atoms of carbon-12. Living organisms keep this ratio due to their intake of water, air and nutrients. However, when an organism dies, the amount of carbon-14 decreases exponentially due to radioactive decay. By measuring the amount that is left, a scientist can estimate the age of something under investigation. This is called **Radiocarbon Dating**. The half-life of carbon-14 is 5,730 years.

Example 4: Determine the model representing the amount Q of carbon-14 remaining t years after the death of an organism.

Solution: Since we know that the half-life of carbon-14 is 5,730, we know that if we start with amount a, then in 5,730 years $\frac{a}{2}$ is left. Putting this into the basic model we have:

$$Q = ae^{bt} \Rightarrow \frac{a}{2} = ae^{b \bullet 5,730} \Rightarrow \frac{1}{2} = e^{5730b} \Rightarrow \ln(.5) = \ln\left(e^{5730b}\right)$$

$$\ln(.5) = 5,730b \Rightarrow b = \frac{\ln(.5)}{5,730} \approx -.000120968$$

Hence our model for carbon-14 dating is $Q = ae^{-.000120968t}$.

Example 5: A museum claims that one of its mummies is 5,000 years old. An analysis reveals that the mummy contains 62% of the expected amount of carbon-14 found in living organisms. Is the museum's claim justified?

Solution: Putting in 5000 years for the time in the model from exercise 4, we have

$Q = ae^{-.000120968(5000)} \Rightarrow Q = ae^{-.60484} \approx a(.55)$ which implies that the mummies should only contain 55% of its original carbon-14. Hence the museum's claim is wrong.

Radiocarbon Dating Model

The quantity Q of carbon-14 remaining t years after the death of an organism (that had an initial amount a) is

$Q = ae^{-.000120968t}$, where b = -.000120968.

Section 7.3: Logarithmic Scales.

A **logarithmic scale** is a scale in which logarithms serve to make data more manageable by expanding small variations and compressing large ones. Two logarithmic scales that are commonly used are the Richter scale, used to rate earthquakes, and the decibel scale, used to rate the loudness of sounds.

We will first look at earthquakes. Most earthquakes are related to **tectonic stress**. A seismograph is an instrument that records the amount of earth movement generated by an earthquake's seismic wave, the recording is called a **seismogram**. The **amplitude** of a seismogram is the vertical distance between the peak or valley of the recording of the seismic wave and a horizontal line formed if there is no earth movement; the amplitude is usually measured in micrometers (μm). It was known that that the amplitude of a recording of a seismic wave is affected by the strength of the earthquake and by the distance between the earthquake and the seismograph.

Richter wanted to develop a scale that that would reflect only the actual strength of the earthquake and not the distance between the **epicenter** and the seismograph. Since larger earthquakes have amplitudes millions of times greater than those of smaller quakes. Richter used the common logarithm of the amplitude of the quake in developing his scale in order to compress the enormous variation inherent in earth movement down to a more manageable range of numbers. Richer study the data from many earthquakes and discovered a pattern. If A_{10} and A_{20} are the amplitudes of one earthquake measured 10 and 20 kilometers, respectively, from the epicenter and if B_{10} and B_{20} are similar 10 and 20 kilometer measurements for a second earthquake, then;

$\log A_{10} - \log B_{10} = \log A_{20} - \log B_{20}$ or $\log \frac{A_{10}}{B_{10}} = \log \frac{A_{20}}{B_{20}}$. This pattern continues regardless of the distance. The only problem is that this scale compared two quakes. So Richter fixed that by creating a “standard earthquake” of a certain fixed strength from which to measure other quakes against. He came up with this “standard earthquake” by averaging a large number of extremely small southern California earthquakes.

Richter’s Definition of Earthquake Magnitude
The magnitude M of an earthquake of amplitude A is $M = \log A - \log A_0$ where A_0 is the amplitude of the “standard earthquake” measured at the same distance.

Example 1: A seismograph 20 kilometers from an earthquake's epicenter recorded a maximum amplitude of 5.0×10^{6} micrometers. Find the earthquake's magnitude.

Solution: $M = \log A - \log A_0 = \log\left(5.0\times10^{6}\right) - \log\left(A_0\right) \approx 6.699 - (-1.7) = 8.399 = 8.4.$

The –1.7 come from the Richter's standard earthquake. So the earthquake's magnitude is 8.4 on the Richter scale.

In addition to comparing the amplitude of an earthquake with that of Richter's artificial "standard earthquake", we can also use Richter's definition to compare the amplitudes of two actual earthquakes.

Magnitude Comparison Formula

If M_1 and M_2 are the magnitudes of two earthquakes, and if A_1 and A_2 are their amplitudes measured at equal distances, then

$$M_1 - M_2 = \log\left(\frac{A_1}{A_2}\right)$$

Example 2: If one earthquake has magnitude 6 and another has magnitude 3, it does not mean that the first caused twice as much earth movement. Find how the stronger earthquake actually compares to the weaker one.

Solution: We use the Magnitude Comparison Formula

$$M_1 - M_2 = \log\left(\frac{A_1}{A_2}\right) \Rightarrow 6 - 3 = \log\left(\frac{A_1}{A_2}\right) \Rightarrow 3 = \log\left(\frac{A_1}{A_2}\right) \Rightarrow 10^3 = \frac{A_1}{A_2}.$$

$$A_1 = 10^3 A_2$$

So the stronger earthquake is actually 1000 times that of the weaker earthquake.

The amount of ground movement of a quake does not depend exclusively on the amount of energy radiated by the quake, because different geological compositions will transmit the energy in different ways. However, ground movement and energy are closely related.

Energy Formula

The energy E (in ergs) released by an earthquake of magnitude M is approximated by:

$$\log E \approx 11.8 + 1.45M$$

Example 3: Shortly after the 1989 San Francisco quake, it was announced that the quake was of magnitude 7.0. Later, after the data from more seismographs had been analyzed, the rating increased to 7.1. How large an increase of energy released corresponds to this increase in magnitude?

Solution: Using the energy formula, and calculating the energy for both magnitudes we have: $\log E_1 \approx 11.8 + 1.45M = 11.8 + 1.45(7.0) = 21.95 \Rightarrow E_1 = 10^{21.95}$

and $\log E_2 \approx 11.8 + 1.45M = 11.8 + 1.45(7.1) = 22.095 \Rightarrow E_2 = 10^{22.095}$.

Hence we have $\dfrac{E_2}{E_1} \approx \dfrac{10^{22.095}}{10^{21.95}} = 10^{22.095-21.95} = 10^{.145} \approx 1.39$,

1.39 rounded is 1.4, hence (1.4) $E_1 \approx E_2$, implying that the second earthquake released 1.4 times more energy that the first estimate.

A sound is a vibration received by the ear and processed by the brain. The **intensity of the sound** is a measure of the "strength" of the vibration. The vibration is determined by placing a surface in the path of the sound and measuring the amount of energy in that surface per unit of area per second. This surface acts like an eardrum. To determine loudness as humans perceive it, we need to look at the ratio of sound intensities. The brain processes sound in a roughly logarithmic fashion.

The **decibel scale** approximates loudness as perceived by the human brain. It is based on the ratio of intensities of sound, and its range is roughly on a par with the range in loudness that a human perceives.

Decibel Rating Definition

If a sound has intensity I in watts per square centimeter, measured at a standard distance, then its decibel rating is

$$D = 10\log\left(\frac{I}{I_0}\right)$$

where I_0 is a "standard intensity" ($I_0 \approx 10^{-16}$ watts/cm^2, the intensity of a barely audible sound).

The word decibel is abbreviated dB. One decibel is one-tenth of a bel.

Example 4: When two students started a conversation in the library, the intensity shot up to $10^{-10}\, watts/cm^2$. Find the increase in decibels when the initial decibel rating was 40 dB.

Solution: So we need to find the decibel rating for when the two students began to talk. So we have the following:

$$I_2 = 10^{-10}; D_2 = 10\log\left(\frac{I_2}{I_0}\right) = 10\log\left(\frac{10^{-10}}{10^{-16}}\right) = 10\log\left(10^{-10-(-16)}\right)$$

$= 10\log\left(10^6\right) = 10 \bullet 6 = 60\text{dB}$, hence the difference is 60 - 40 = 20 dB.

There is a slightly different formula that can be used to find the decibels.

dB Gain Formula

If I_1 and I_2 are the intensities of two sounds, then the dB gain is

$$D_1 - D_2 = 10\log\left(\frac{I_1}{I_2}\right).$$

Example 5: The noise on a busy freeway varies from 81 dB to 92 dB. Find the corresponding variation in intensities.

Solution: We are given

$D_1 = 92$ dB and $D_2 = 81$ dB and we are asked to compare I_1 and I_2. Hence we have

$$D_1 - D_2 = 10\log\left(\frac{I_1}{I_2}\right) \Rightarrow 92 - 81 = 10\log\left(\frac{I_1}{I_2}\right) \Rightarrow 11 = 10\log\left(\frac{I_1}{I_2}\right)$$

$$\Rightarrow 1.1 = \log\left(\frac{I_1}{I_2}\right) \Rightarrow 10^{1.1} = \frac{I_1}{I_2} \Rightarrow I_1 \approx 12.5893 I_2.$$

The End!