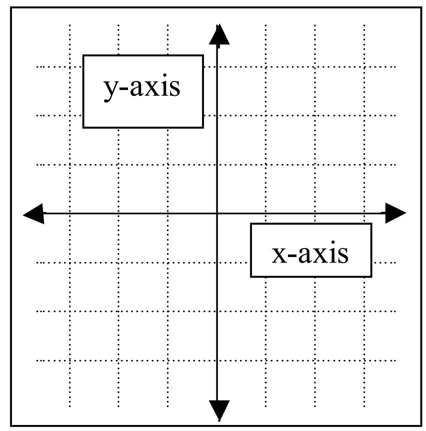
Chapter 3

Section 3.1: Graphing Lines in the Coordinate Plane

In this chapter we will learn about the Cartesian coordinate system. Understanding the Cartesian coordinate system is essential for developing higher mathematics skills. The Cartesian system is used to draw **graphs** with. Graphs show relationships between variables in an equation by letting us draw pictures of the relation. We are familiar with the regular real number line. Each point on the real number line corresponds to a single real number. We call this single real number a **coordinate**. As I have explained before, the usual real number line is what is called **one-dimensional**. That is because you can move in one direction either forward, which is a positive direction, or backward , which is a negative direction. We live in a three dimensional world. We are able to see things in three dimensions. That is, things have a volume, which is based on an objects height, width and length. So, if we live in a three-dimensional world, then does two dimensions exist? Yes.



We measure **two dimensions on a Cartesian plane** using **Cartesian coordinates**. The Cartesian plane is two-dimensional because we take two real number lines and **put** them together to form the Cartesian plane. We take a horizontal number line, then we cut the horizontal number line with a vertical number line. We call the horizontal number line the **x-axis**, and the vertical number line the **y-axis.** Hence, we get a **two dimensional plane**. That is, there are two ways you can move, **horizontally (forward/backward)**, and in a **vertical direction (up/down)**.

Up goes in a positive direction on the vertical number line, and down goes in a negative direction on the vertical number line.

Since the Cartesian plane has two directions, which means it is two dimensional, we have to have two numbers associated with each point in the Cartesian plane. The first number of the point tells you how many units on the x-axis you move (forward/backward). The second number tells you how many units on the y-axis you move (up/down). When we put these two numbers together we get a **Cartesian coordinate in two dimensions**. If we let **a** and **b** be two real numbers, we denote the point described in two dimensions by **a** and **b** as (**a**, **b**), which is called an **ordered pair**. So, this tells me I move **a** units on the x-axis (horizontal direction) and **b** units on the y-axis. If **a** is positive, then I move forward (to the right) on the x-axis. If **a** is negative, I move backwards (to the left) on the x-axis. If **b** is positive, I move up on the y-axis. If **b** is negative, I move down on the y-axis. We call **a** the **x-coordinate** and **b** the **y-coordinate**. With these different directions, there are four possible combinations we can get from **a** and **b**. Hence, we divide the Cartesian plane up into four **quadrants**. Each quadrant depends on if **a** and **b**, are positive or negative. The four quadrants are defined as follows:

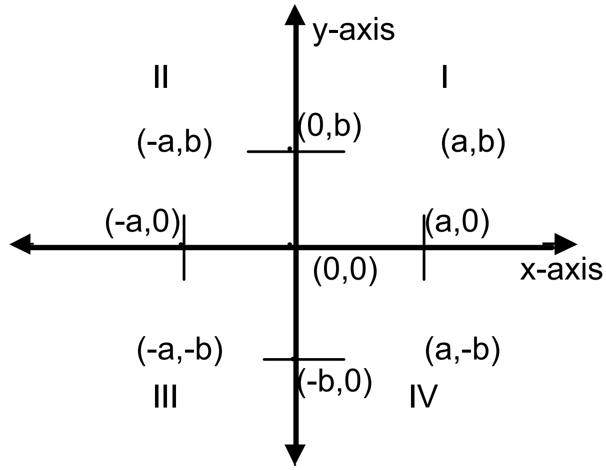
<u>Quadrant I</u>: This is the upper right corner of the Cartesian plane. In this quadrant both the x-coordinate and the y-coordinate are positive. So the general form of a Cartesian coordinate in the first quadrant is of the form (**a**, **b**), where **a** > 0 and **b** > 0.

<u>Quadrant II</u>: This is the upper left corner of the Cartesian plane. In this quadrant the x-coordinate is negative and the y-coordinate is positive. So the general form of a Cartesian coordinate in the second quadrant is of the form (- **a**, **b**), where - **a** < 0 and **b** > 0.

<u>Quadrant III</u>: This is the lower left corner of the Cartesian plane. In this quadrant both the x-coordinate and y-coordinate are negative. So the general form of a Cartesian coordinate in the third quadrant is of the form (- **a**, - **b**), where - **a** < 0 and - **b** < 0.

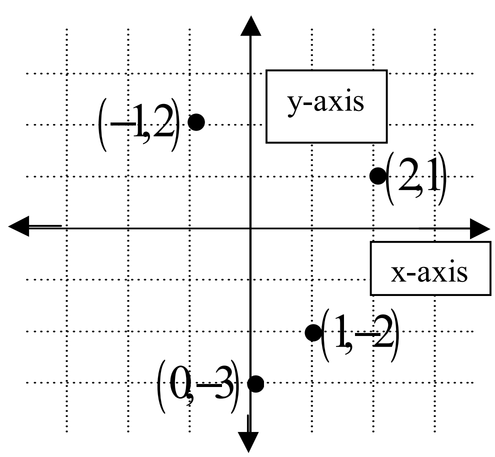
<u>Quadrant IV</u>: This is the lower right corner of the Cartesian plane. In this quadrant the x-coordinate is positive and the y-coordinate is negative. So the general form of a Cartesian coordinate in the forth quadrant is of the form (**a**, - **b**), where **a** > 0 and - **b** < 0.

There are two further observations that need to be made. If the y-coordinate is 0, that is **b** = 0, so that we have **(a, 0), the point lies on the x-axis**, **a** is either positive or negative. Notice that the Cartesian coordinate is now a one-dimensional point. If the x-coordinate is 0, that is **a** = 0, so that we have **(0, b), the point lies on the y-axis**, **b** is either positive or negative, and again the point is a one-dimensional point. If both **a** and **b** are 0, that is we have the point (0, 0) we are at the place where the x-axis and the y-axis intersect, that is cross, and this is called the **origin**.



Example 1: Plot the following points on the Cartesian plane, (-1,2), (0,-3), (1,-2),(2,1).

To plot these points, the first number tells you how many units in the horizontal direction to move, the second number tells you how many units in the vertical direction to move. Hence I have the following diagram:



A **linear equation** in two variables is an equation that can be put in the form:

$$ax + by = c$$

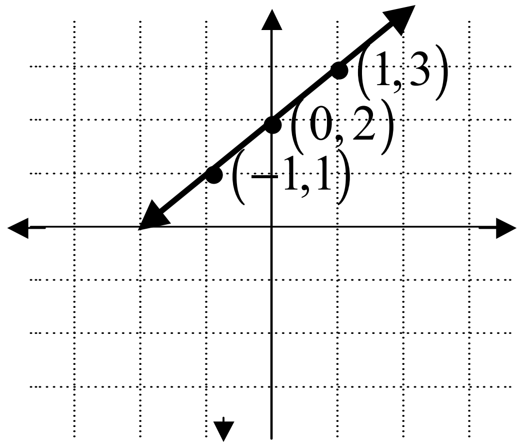
where a, b, and c are real numbers.

The above equation type is called **linear**, because the graph of an equation of that form is a straight line. A line in the form $ax + by = c$ is said to be in **standard form**. Now, we are going to test if an ordered pair lies on a line. If the ordered pair lies on a line, when you substitute the ordered pair into the equation of the line, you should get a true statement. In other words, when you substitute the values from the ordered pair for x and y in the equation, you should produce a true statement. If a set of points are on the same line, the points are called **collinear**. A **graph** of an equation is an illustration of a set of points whose coordinates satisfy the equation.

We learned above that an equation of the form $ax + by = c$ is a straight line. One way to graph the line in the Cartesian plane is by **plotting points**.

Example 3: Graph the equation $2y = 2x + 4$ by plotting points.

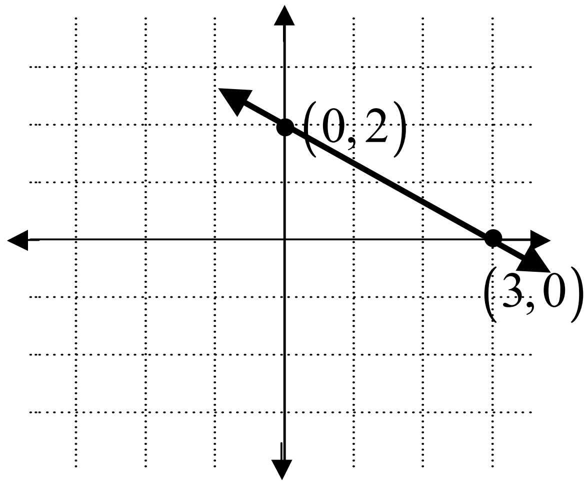
Solution: First we need to solve the equation for y. Hence, dividing through by 2 we have $y = x + 2$. Now we need to pick three values for x, I will choose 0, 1 and –1. For $x = 0$, I get $y = 0 + 2 = 2$, so the ordered pair (0,2) is on the line. For $x = 1$, I get $y = 1 + 2 = 3$, so the ordered pair (1,3) is on the line. For $x = -1$, I get $y = -1 + 2 = 1$, so the ordered pair (-1,1) is on the line. Plotting these points I have the following:



Actually you only need two points to draw a straight line. Earlier we talked about the two points the ***x* – intercept** (*x*, 0), and the ***y* – intercept** (0, *y*). If we can find these two points for a line, then we can draw a straight line through them and get the graph of the line. To find the *x* – intercept, we set the *y* variable equal to zero and solve for *x*. To find the *y* – intercept, we set the *x* variable equal to zero and solve for *y*. Then we plot these two points and connect them with a straight line and that gives us the graph of the line.

Example 4: Graph the line by finding the intercepts $-3y - 2x = -6$.

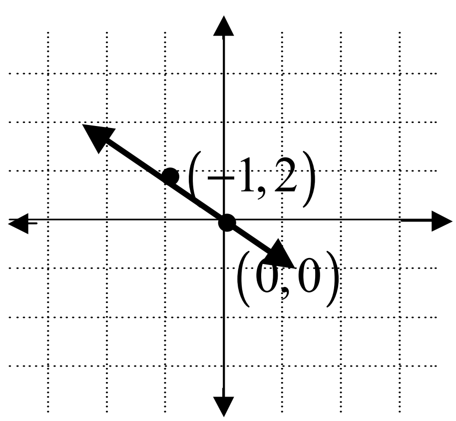
Solution: To find the *x* – intercept (*x*, 0), this tells me to set $y = 0$ and solve for *x*. Here we have $-3(0) - 2x = -6$. Hence, I get $-2x = -6$, so $x = -6/-2 = 3$. So the *x* – intercept is (3, 0). To find the *y* – intercept (0, *y*), this tells me to set $x = 0$ and solve for *y*. Hence, I get $-3y - 2(0) = -6$, so $-3y = -6$ and I get $y = -6/-3 = 2$. So the *y* – intercept is (0, 2). Plotting these two points and drawing an arrow through them I get the following graph:



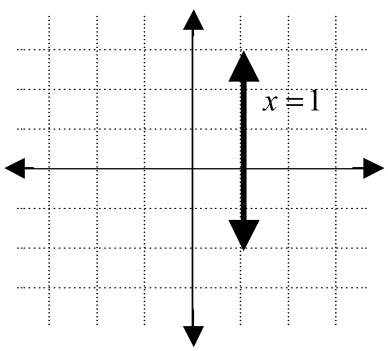
Some lines have the equation $ax + by = 0$. This just means that the constant is equal to zero. These lines have graphs that pass through the origin of the Cartesian plane.

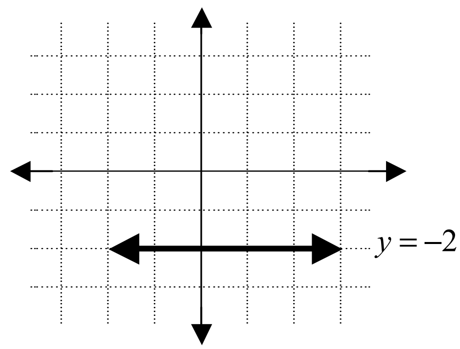
Example 5: Graph the equation $4x + 2y = 0$.

Solution: Notice that this graph has the form $2y = -4x$. So we have the line $y = -2x$. Since the x – intercept and the y – intercept are the same point, (0, 0), we need one more point to graph this line. Choosing $x = -1$, I have that $y = -2(-1) = 2$. So the ordered pair (-1, 2) is on the graph. Plotting these two points we have the following graph:



Equations of horizontal and vertical lines are different from the previous lines we have looked at. If the linear equation has only one variable, then the line is either vertical or horizontal. A **vertical line** has the equation of the form x = **a**, where **a** is a real number. This means that for any value of y, x will be the same value as y varies. A **horizontal line** has the equation of the form y = **b**, where **b** is a real number. This means that as x varies, y will be the same value.





<u>Section 3.2</u>: The slope of a Line

Now we are going to look at the equations of straight lines. The slope of a line measures the relationship between two directions. The directions we called the rise and the run. The **rise** can now be officially called the **change in the vertical or y-axis direction**. The **run** is the **change in the horizontal or x-axis direction**. Recall, to find the "change" in each direction we just subtract the coordinates in the direction you want to find the change in. The slope turns out to be the ratio of the two "change" numbers. The change is denoted by a special notation. This notation is called **delta.** So the change in the x-axis direction is called **delta *x***, the change in the y-axis direction is called **delta *y***. Delta x is denoted by Δx. Delta y is denoted by Δy. So, we use a little triangle in front of our variables to signify that we are measuring the change or difference in that particular direction. The triangle is called delta. We usually use the letter m to stand for slope. So, in general the slope of a line is the equation:

$$\text{slope} = \text{m} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}, \Delta x \neq 0.$$

When we have two points in the plane, these two points determine a straight line. In order to calculate the slope of a line, we must have two points on the line. Also, no matter which two points on the line you use, the slope will be the same. For a general line, let p_1 denote the first point on the line, and p_2 denote the second point on the line. Then, the points have the coordinates: $p_1 = (x_{1,}y_1)$ and $p_2 = (x_2, y_2)$. Hence, we have for the slope of a general line:

$$\text{slope} = \text{m} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{(y_2 - y_1)}{(x_2 - x_1)}, x_2 - x_1 \neq 0 \text{ or } x_2 \neq x_1 .$$

Observation: We have to include the condition that $x_2 - x_1 \neq 0$, because if this difference was zero, then we would be dividing by zero. **Recall, we never divide by zero!** If it is true that this difference is zero, then we know that $x_2 = x_1$. We will see in a little while that when this happens, we have a vertical line. What do you think this tells us about the slope of a vertical line?

Example 4: Find the slope of the line which has the following two points on it: (4, 6) and (5, 8).
So our x-coordinates are 4 and 5. Our y-coordinates are 6 and 8. So the **run** will be the difference between the x-coordinates or 4 and 5. The **rise** will be the difference between the
y-coordinates or 5 and 8. Remember, it does not matter the order in which you subtract the numbers, as long as you are consistent. I�ll calculate this slope both ways below.
First, letting $p_1 = (4,6)$ and $p_2 = (5,8)$, where $y_2 = 8,\ y_1 = 6,\ x_2 = 5,\ x_1 = 4$, we have for the slope:

$$\text{slope} = \text{m} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(8-6)}{(5-4)} = \frac{2}{1} = 2 .$$

Second, letting $p_1 = (5,8)$ and $p_2 = (4,6)$, where $y_2 = 6, y_1 = 8, x_2 = 4, x_1 = 5$, we have for the slope:

$$\text{slope} = \text{m} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(6-8)}{(5-4)} = \frac{-2}{-1} = \frac{2}{1} = 2 .$$

Remember, the negatives cancel each other out. *So, what the slope $2 = \frac{2}{1}$ tells me, is that if I move two units up in the vertical or y-axis direction,* I *must move one unit over in the horizontal or x-axis direction, <u>in order to stay on the line that contains the two points (4, 6) and (5, 8).</u>*

If the slope is negative, then either $\Delta y < 0$ or $\Delta x < 0$. If the slope is negative, the line slants to the left. If the slope is positive, the line slopes to the right. Please review that part of the lesson to understand these two cases.

I said earlier that if we had that $x_2 = x_1$ which implies that $\Delta x = 0$, we have a **vertical line**. Since the slope is the ratio of the rise divided by the run, in this case we have no run. That is, there is no movement in the horizontal or x-axis direction. So the only movement is the rise, which is in the vertical or y-axis direction. Hence the line is vertical.

Caution: since we are dividing the rise by a zero, **the slope of a vertical line does not exist!**

On the other hand, if $y_2 = y_1$ or $\Delta y = 0$, we have no rise. So in the ratio of the rise divided by the run, we have no rise. We have a zero in the numerator of the slope's ratio. When that happens, the slope is equal to zero. So we have no movement in the vertical or y-axis direction. But, we have movement in the horizontal or x-axis direction. Hence, we have a **horizontal line**. **BUT**, the **slope of a horizontal line does exist and is equal to zero.** There is a BIG difference here! When a zero is in the numerator of a fraction, the fraction is equal to zero. But, when there is a zero in the denominator of a fraction, the fraction does not exist or rather is not defined.

Section 3.3: Three Forms for the Equation of a Line

From the equation for the slope of a line, we can obtain the general equations for a straight line.

$$\text{m} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$ where $\Delta x \neq 0$. Multiply both sides by the denominator

to get;

$(x_2 - x_1)m = (y_2 - y_1)$ OR $(y - y_1) = m(x - x_1)$, which is called the **Point-Slope form** of the equation of a line.

We use the point-slope form equation to obtain another type of equation for straight lines. This form for the equation of a line requires that we know the **y-intercept**. Recall, that the y-intercept is the place that the graph crosses the y-axis. So, the point at which the y-intercept occurs has the form (0, **b**), where **b** is a number. So, putting in this point in the point-slope form where $y_1 =$ **b** and $x_1 = 0$ we have the following:

$y_2 - b = m(x_2 - 0)$ or $y_2 - b = mx_2$, solve for y_2 to get; $y_2 = mx_2 + b$.

We usually drop the subscript 2 to get the following **slope-intercept form** for the equation of a line: $y = mx + b$.

Observation: To obtain the slope-intercept form for the equation of a line, we could also just use the formula for the slope. Both the point-slope and slope-intercept forms of the equations for lines can both be obtained by the equation for the slope. There **is not** three equations to memorize here. You can derive each of the forms for the equation of a line by using the formula for the slope. For example, lets derive the slope-intercept form via the formula for the slope. Using the point (0, **b**), where **b** is a number, as our y-intercept, we have the following: let $y_1 = b$ and $x_1 = 0$ put this information into the equation for the slope of a line, then solve for y_2.

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(y_2 - b)}{(x_2 - 0)} = \frac{(y_2 - b)}{x_2};\ x_2 \neq 0.$$ Solving for y_2 we get

$mx_2 = (y_2 - b)$ so $y_2 = mx_2 + b$. So we have $y = m\,x + b$.

Example 5: Graph $9x - 3y = 9$.
Step 1: We need to find out what the equation is. Divide the equation through by 3 to get:

$$3x - y = 3.$$

Step 2: If we solve for y, then we will have the slope-intercept form for the equation of a line:
$3x - 3x - y = 3 - 3x$

$$0 - y = 3 - 3x$$

$-y = 3 - 3x$ Multiply through by -1 so that the coefficient of y is 1.

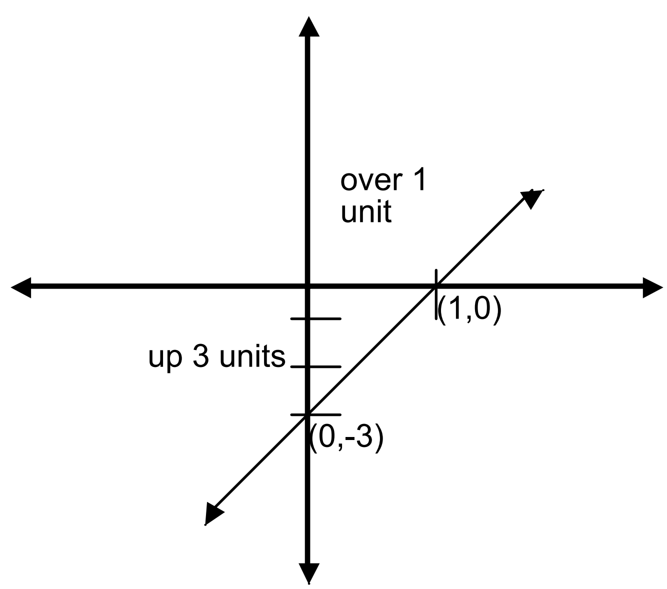
$$y = -3 + 3x \quad \text{OR} \quad y = 3x - 3.$$

Step 3: By the last equation of step 2, we know that this line has a slope of 3, or $\frac{3}{1}$. So for each one unit in the horizontal or x-axis direction I move, I will move 3 units up in the vertical or y-axis direction. However, I need to know at least one point on the line in order to graph the line. I really don't need to know two points, since I know what the slope is. So, the easiest point to find is the y-intercept. To do this, set $x = 0$ and solve for y. So we have the following:

$$y = 3(0) - 3 = -3.$$

So the y-intercept is the point (0, -3). Now we can also find the x-intercept by setting $y = 0$ and solving for x. The x-intercept is the point (1, 0).

Step 4: Now there are two ways we can graph the line. We can plot the y-intercept, then count over 1 horizontally then three units up vertically from the y-intercept to find another point on the line, and then draw a straight line through the two points that slants to the right since the slope is positive. Or we can plot both the y-intercept and the x-intercept then draw a straight line between the points that slants to the right.



Example 6: Find the equation of a line that goes through the points (2, -3) and (6, -5).
Step 1: In order to find the equation of a line, we must first find the slope of the line.

slope = m = $\frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(-5-(-3))}{(6-2)} = \frac{(-5+3)}{(6-2)} = \frac{-2}{4} = -\frac{1}{2}$. Step 2: Using one of the given points and the slope we found, we can use the point-slope form for the equation of a line in order to find the equation for the line. Using the point (2, -3) where $x_1 = 2$ and $y_1 = -3$ we have the following:

$$(y_2 - (-3)) = \left(-\frac{1}{2}\right)(x_2 - 2)$$

Dropping the subscripts and simplifying we have the following:

$$y + 3 = (-\frac{x}{2}) + 1$$

$$\left(\frac{1}{2}\right)x + y + 2 = 0 \Rightarrow x + 2y + 4 = 0$$

The last equation is called the **general form/standard form** for the equation of a line. In this form we just put all values and variables on one side of the equation and set the equation equal to zero.

Observation: Do you know what a line is? Well, the last equation in example 6 gives us the equation for a line. This equation means, that when you graph the line, all the points on that line will satisfy the equation $\frac{1}{2}x + y + 2 = 0$. For example, the point (6, -5) is on that line, so putting those values in for x and y, we get $\frac{1}{2}(6) + (-5) + 2 = 3 - 5 + 2 = 0$.

Hence, any point on the line will do the same thing.

Summary: The slope of a line tells us how many units we need to move in both the vertical and horizontal directions in order to remain on the line. The equation for the slope is just the difference of the y-coordinates divided by the difference of the x-coordinates. This is called the **rise** divided by the **run.** If the **run is 0**, the **slope does not exist**, so we have a **vertical line**. If the **rise is 0**, then **the slope is 0**, so we have a **horizontal line**. A line is a graphical representation of all the sets of points whose values for x and y will satisfy the equation given by the general form for the equation of a line.